Math 564: Advance Analysis 1
Lecture 12

Integration of simple functions. Let $(X, \mathcal{M}, \mu)$ be a measure space. We define the μ-integral of a simple function $f \in L := L(X, \mathcal{M}) := \{\mathcal{M}\text{-measurable functions } X \to \mathbb{R}\}$ by

$$\int f \, d\mu := \sum_{i<n} a_i \cdot \mu(B_i),$$

where $f = \sum_{i<n} a_i \mathbb{1}_{B_i}$ is some/any presentation.

For $B \in \mathcal{M}$, put $\int_B f \, d\mu := \int f \cdot \mathbb{1}_B \, d\mu$.

Prop. This is well-defined (i.e. doesn't depend on the choice of presentation).

Proof. HW.

Properties. Let $f, g \in L := L(X, \mathcal{M})$ be simple functions.

(a) Non-negativity: $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$. Hence, $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(b) Linearity: $\int (af + g) \, d\mu = a \cdot \int f \, d\mu + \int g \, d\mu$ for all $a \in \mathbb{R}$.

(c) If $f \geq 0$, then $\int f \, d\mu = 0 \iff f = 0$ a.e.

(d) If $f \geq 0$, then the map $B \mapsto \int_B f \, d\mu$ is a measure on $\mathcal{M}$ denoted by μ_f.

Proof. (a) is by definition, (b) is by well-definedness.
For (c), we use that $0 \cdot \infty = 0$.
For (d), we only need to show ctbl additivity, so let $B = \bigsqcup_{n \in \mathbb{N}} B_n$ in $\mathcal{M}$.
Fix a representation $f = \sum_{i<m} a_i \mathbb{1}_{C_i}$. Then

$$\mu_f(B) := \int_B f\,d\mu = \int f\cdot \mathbb{1}_B\,d\mu = \sum_{i\le m} a_i\,\mu(C_i\cap B) \overset{\text{ctbl add. of }\mu}{=} \sum_{i\le m} a_i \sum_{n\in\mathbb{N}} \mu(C_i\cap B_n) =$$

$$= \sum_{n\in\mathbb{N}}\sum_{i\le m} a_i\,\mu(C_i\cap B_n) = \sum_{n\in\mathbb{N}} \int \sum_{i\le m} a_i \mathbb{1}_{C_i}\cdot \mathbb{1}_{B_n}\,d\mu = \sum_{n\in\mathbb{N}} \int_{B_n} f\,d\mu$$

$$= \sum_{n\in\mathbb{N}} \mu_f(B_n).$$

$\square$

<u>Integration of L^+.</u> Let $(X,\mathcal{M},\mu)$ be a measure space. We learn how to approximate L^+ functions from below by simple functions.

<u>Recall notions of convergence.</u> For a set X and a metric space (Y,d) (for example $Y=\mathbb{R}$). Let $f_n : X\to Y$ and $f: X\to Y$.

- We say that (f_n) converges to f pointwise, and write $f_n \to f$ ptwise, if for each $x\in X$, $f_n(x)\to f(x)$ in the top. of Y.
- We say that (f_n) converges to f uniformly, and write $f_n \rightrightarrows_u f$, if $d_u(f_n,f)\to 0$ as $n\to\infty$, where for $f,g: X\to Y$

$$d_u(f,g) := \sup_{x\in X} d(f(x),g(x)),$$

we call this the uniform distance between f and g.
When $Y=\mathbb{R}$, we also write $\|f\|_u := \sup_{x\in X}|f(x)|$, so $d_u(f,g)=\|f-g\|_u$, and we call $\|\cdot\|_u$ the uniform norm.

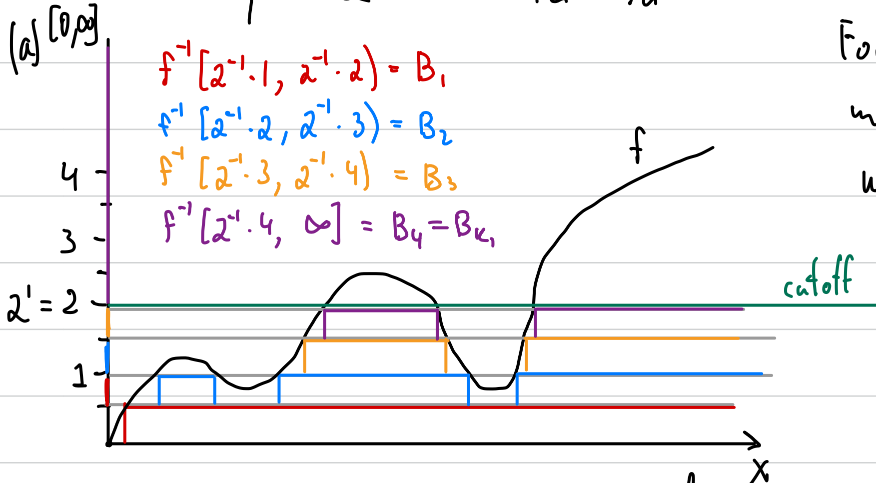
<u>Prop.</u>

(a) For every $f\in L^+$, there is a sequence $(f_n)\subseteq L^+$ of simple functions such that $f_0\le f_1\le f_2\le \dots \le f$, $f_n\to f$ pointwise on X, the convergence is uniform on every set $X'\subseteq X$ on which f is bounded.

(b) For every $f\in L$, there is a sequence $(f_n)\subseteq L$ of simple functions s.t. $|f_0|\le |f_1|\le \dots \le |f|$, $f_n\to f$ pointwise, and $f_n|_{X'} \rightrightarrows_u f|_{X'}$ for each $X'\subseteq X$ on which f is bounded.

Proof. (b) follows from (a) by writing $f = f^+ - f^-$, getting sequences (f_n^+) and (f_n^-) of simple functions satisfying (a) for f^+ and f^-, then the sequence $f_n := f_n^+ - f_n^-$ is as desired.

(a)



For each n, we will try to approximate the cutoff of f at 2^n, i.e. $\min(f, 2^n)$. We partition the codomain $[0, 2^n]$ into intervals of equal length 2^{-n}, so the total of $k_n := 2^n/2^{-n} = 2^{2n}$.

$B_k := f^{-1}[2^{-n}\cdot k, \infty]$, $k = 1, \dots, k_n$.

Note that $B_1 \supseteq B_2 \supseteq \dots \supseteq B_{k_n}$ and put $f_n := \sum_{k=1}^{k_n} 2^{-n}\cdot \mathbb{1}_{B_k}$. Note that the standard presentation of f_n is $f_n = \sum_{k=1}^{k_n} 2^{-n} k \cdot \mathbb{1}_{B'_k}$, where $B'_k := B_k \setminus B_{k+1}$ for $1 \le k < k_n$ and $B'_{k_n} := B_{k_n}$.

Putting $X_n := f^{-1}[0, 2^n]$, we see that $\| f|_{X_n} - f_n \| \le 2^{-n}$ and the X_n increase. Thus if f is bounded on X', then $X' \subseteq X_m$ and we have $\| f|_{X'} - f_n|_{X'} \| \le 2^{-n}$ for all $n \ge m$, so $f_n|_{X'} \to_u f|_{X'}$.

In particular, for any $x \in X$ with $f(x) < \infty$, $f_n(x) \to f(x)$.

And for $x \in X$ with $f(x) = \infty$, one verifies by inspection that $f_n(x) = 2^n \to \infty$. $\square$

Now it's reasonable to define the μ-integral of $f \in L^+$ by

$$\int f \, d\mu := \sup \left\{ \int s \, d\mu : s \in L^+ \text{ simple and } s \le f \right\}.$$

Note that if f itself is simple, the two definitions of $\int f \, d\mu$ coincide because $\int f d\mu$ will be accounted for in the sup.

Observations. Let $f, g \in L^+$.

(a) Non-negativity: $\int f \, d\mu \ge 0$. In particular, $f \le g \Rightarrow \int f \, d\mu \le \int g \, d\mu$.

(b) $\int f \, d\mu = 0 \iff f = 0$ a.e.

(c) Scaling: $\int a\cdot f\,d\mu = a\cdot\int f\,d\mu$ for all $a\in[0,\infty)$.

Proof. We only prove (c).

$\Rightarrow$. Suppose $\int f\,d\mu=0$. Then if f was not 0 a.e., there would be some $n\in\mathbb{N}^+$ s.t. $f\geq\frac{1}{n}$ on a positive measure set B_n.
So the simple function $s=\frac{1}{n}\cdot\mathbb{1}_{B_n}$ is $\leq f$ and $\int s\,d\mu=\frac{1}{n}\cdot\mu(B_n)$ which is positive, a contradiction.

$\Leftarrow$. If $f=0$ a.e., then any simple function $0\leq s\leq f$ is also 0 a.e. so $\int s\,d\mu=0$. $\square$

Note that to compute $\int f\,d\mu$, we would to approximate f from below by simple functions (f_n) and hope that $\int f\,d\mu=\lim_n\int f_n\,d\mu$.
Also, we would like to prove linearity of $\int$, which amounts to finite additivity of $\int$. All this follows from:

Monotone Convergence Theorem. Let $f_n, f\in\mathcal{L}^+$ and $f_n\nearrow f$, i.e. $f_0\leq f_1\leq f_2\leq\ldots\leq f$ and $f_n\to f$ ptwise. Then $\int f_n\,d\mu\nearrow\int f\,d\mu$.

Proof. Let $f_n\nearrow f$ ptwise. Then $\int f_n\,d\mu\leq\int f\,d\mu$, so $\lim_n\int f_n\,d\mu = \sup_n\int f_n\leq\int f\,d\mu$. We prove the converse $\lim_n\int f_n\,d\mu\geq\int f\,d\mu$,

for which we fix a simple function $0\leq s\leq f$ and show that $\lim_n\int f_n\,d\mu\geq\int s\,d\mu$. We give ourselves an ε room, and show that

$$\lim_n\int f_n\,d\mu\geq(1-\varepsilon)\cdot\int s\,d\mu=\int(1-\varepsilon)s\,d\mu$$

for all $\varepsilon>0$. Let $X_n:=\{x\in X: f_n(x)\geq(1-\varepsilon)s\}$. Then $X=\bigcup_{n\in\mathbb{N}}\uparrow X_n$,
so $\int f_n\,d\mu\geq\int_{X_n}f_n\,d\mu\geq\int_{X_n}(1-\varepsilon)s\,d\mu$ and by upward monotonicity of the measure $\mu_{(1-\varepsilon)s}$,

we have that $\sup_n \int_{X_n} (1-\varepsilon) s \, d\mu = \int (1-\varepsilon) s \, d\mu$, hence

$\lim_n \int f_n \, d\mu \geq \int (1-\varepsilon) s \, d\mu$. $\square$